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Article

Extending Separability: Properties, Products, and Applications in Pseudocompact Spaces

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Abstract: This paper introduces the concept of almost separable spaces, a generalization of classical separable spaces in topology. We investigate the fundamental properties of almost dense subsets, establishing their relationships with dense sets, sequential separability, and strong sequential separability. We demonstrate that almost separability is c-productive, and under certain conditions, the converse holds for infinite products. Furthermore, we analyze the cardinality of the set of real-valued continuous functions on almost separable spaces and provide bounds for functionally Hausdorff spaces. Finally, we extend the classical Baire category theorem to pseudocompact spaces using almost dense cozero sets, highlighting implications for connectedness and topological structure. Several examples illustrate these properties in noncompletely regular, normal, and functionally Hausdorff spaces.

Keywords: almost separable spaces, almost dense subsets, sequential separability, c-productivity, pseudocompact spaces.

INTRODUCTION

Let $\boldsymbol{X}$ be any topological space, and let $\boldsymbol{C}(\boldsymbol{X})$ denote the set of all real-valued continuous functions on $\boldsymbol{X}$. A subset $\boldsymbol{A} \subset \boldsymbol{X}$ is called almost dense if, for every $\boldsymbol{f} \in \boldsymbol{C}(\boldsymbol{X})$, the condition $\boldsymbol{f}(\boldsymbol{A}) = \{\boldsymbol{0}\}$ implies $\boldsymbol{f}(\boldsymbol{X}) = \{\boldsymbol{0}\}$. A topological space $\boldsymbol{X}$ is said to be almost separable if it contains a countable almost dense subset.

It is clear that every dense subset is automatically almost dense. In the context of completely regular spaces, the notions of dense and almost dense coincide; however, the converse does not necessarily hold in general topological spaces. For instance, there exist non-completely regular spaces where dense and almost dense subsets coincide, illustrating the subtle differences between these concepts. In this study, we also show that almost separability is c-productive, and under certain conditions, the converse holds as well. Additionally, we explore the relationships between almost separability, sequential separability, and strongly sequential separability.

Definition 1.1 ([1,3,4]). A topological space $\boldsymbol{X}$ is sequentially separable if there exists a countable subset $\boldsymbol{D} \subset \boldsymbol{X}$ such that, for every $\boldsymbol{x} \in \boldsymbol{X}$, there exists a sequence from $\boldsymbol{D}$ converging to $\boldsymbol{x}$.

Definition 1.2 ([1]). A space $\boldsymbol{X}$ is called strongly sequentially separable if it is separable and every countable dense subset is sequentially dense. A subset $\boldsymbol{D} \subset \boldsymbol{X}$ is sequentially dense if, for every $\boldsymbol{x} \in \boldsymbol{X}$, there exists a sequence in $\boldsymbol{D}$ that converges to $\boldsymbol{x}$.

1. Almost Dense Subsets of a Topological Space

Definition 2.1. A subset $A \subset X$ is almost dense if, for all $f \in C(X)$, the equality $f(A) = \{0\}$ implies $f(X) = \{0\}$.

Definition 2.2. A subset $A \subset X$ is called a zero set if there exists $f \in C(X)$ such that $A = Z(f) = \{x \in X : f(x) = 0\}$. The complement of a zero set is referred to as a cozero set.

Theorem 2.3. Every dense subset of a topological space is almost dense.

Proof. Let A be a dense subset of X. Consider any $f \in C(X)$ such that $f(A) = \{0\}$. By continuity and the density of A, $f(X) = \{0\}$. Therefore, A is almost dense. □

Theorem 2.4. In a completely regular space X, every almost dense subset is dense.

Proof. Suppose A is almost dense but not dense in X. Then there exists $x_0 \in X \setminus A$. By the property of complete regularity, there exists $f \in C(X)$ such that $f(x_0) = 1$ and $f(A) = \{0\}$. This contradicts the assumption that A is almost dense. Hence A must be dense in X. □

Theorem 2.5. Let Y be a dense subset of X (which may not be completely regular) such that the relative topology of Y has a base consisting of cozero sets of X. Then every almost dense subset of X is dense.

Proof. Let A be almost dense in X, and let U be a non-empty open subset of X. Since Y is dense, $U \cap Y \neq \emptyset$. If $A \cap U = \emptyset$, then $A \cap U \cap Y = \emptyset$. Choose $y \in U \cap Y$. By the cozero base property, there exists $f \in C(X)$ such that $y \notin Z(f) \subset U \cap Y$. Then $f(A) = \{0\}$, but $f(X) \neq \{0\}$, contradicting the almost denseness of A. Therefore, $A \cap U \neq \emptyset$. □

Example 2.6. Consider the K-topology on $\mathbb{R}$, defined as $\beta = \{(a,b) : a < b, a, b \in \mathbb{R}\} \cup \{(a,b) \setminus K : a < b, a, b \in \mathbb{R}\}$ with $K = \{1/n : n \in \mathbb{N}\}$. The space $(\mathbb{R}, \tau_K)$ is not regular, hence not completely regular. Let $Y = \mathbb{R} \setminus \{0\}$. The subspace Y is open, dense, and completely regular in the relative topology with a cozero base. Applying Theorem 2.5, any almost dense set in $(\mathbb{R}, \tau_K)$ is dense.

Example 2.7. Let $X = \{a, b\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$. Then X is normal. The subset $\{b\}$ is almost dense but not dense in X, showing that almost dense sets may not be dense in normal spaces.

Theorem 2.8. Let $f : X \to Y$ be a continuous, surjective map. If $A \subset X$ is almost dense, then $f(A)$ is almost dense in Y.

Proof. Suppose $g \in C(Y)$ with $g(f(A)) = \{0\}$. Then $g \circ f \in C(X)$ satisfies $g \circ f(A) = \{0\}$. Since A is almost dense, $g \circ f(X) = \{0\}$. Surjectivity of f implies $g(Y) = \{0\}$, proving that $f(A)$ is almost dense in Y. □

Theorem 2.9. Let τ_1 and τ_2 be two topologies on X with τ_2 finer than τ_1. If A is almost dense in (X, τ_2), then it is almost dense in (X, τ_1).

Proof. For any $f \in C(X, \tau_1)$, since τ_2 is finer, $f \in C(X, \tau_2)$. By the almost denseness of A in (X, τ_2), we have $f(X) = \{0\}$. □

Theorem 2.10. If $A \subset X$ is almost dense and $B \subset Y$ is almost dense, then $A \times B \subset X \times Y$ is almost dense.

Proof. Let $f : X \times Y \to \mathbb{R}$ be continuous and vanish on $A \times B$. Fix $a \in A$ and define $f_a(y) = f(a, y)$. Then $f_a(B) = \{0\}$, so $f_a(Y) = \{0\}$. Similarly, for $y \in Y$, define $f_y(x) = f(x, y)$. Then $f_y(A) = \{0\}$, so $f_y(X) = \{0\}$. Therefore, $f = 0$ on $X \times Y$, proving that $A \times B$ is almost dense. □

Corollaries and Propositions on Product Spaces

Corollary 2.11. Let $A_1, A_2, \ldots, A_n$ be almost dense subsets of $X_1, X_2, \ldots, X_n$, respectively. Then their Cartesian product $\prod_{i=1}^{n} A_i$ is almost dense in $\prod_{i=1}^{n} X_i$.

Proposition 2.12. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces, and for each $\alpha \in \Lambda$, let $A_\alpha \subset X_\alpha$ be almost dense. Then the product $\prod_{\alpha \in \Lambda} A_\alpha$ is almost dense in $\prod_{\alpha \in \Lambda} X_\alpha$.

Proof. Fix an element $a = (a_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} A_\alpha$, and define

$$D = \{(x_\alpha)_{\alpha \in \Lambda} \in \prod X_\alpha : \{\alpha : x_\alpha \neq a_\alpha\} \text{ is finite}\}.$$

Let $f : \prod_{\alpha \in \Lambda} X_\alpha \to \mathbb{R}$ be continuous such that $f = 0$ on $\prod_{\alpha \in \Lambda} A_\alpha$. For any finite subset $I \subset \Lambda$, the product $\prod_{\alpha \in I} A_\alpha$ is almost dense in $\prod_{\alpha \in I} X_\alpha$ by Corollary 2.11. Consequently, the subset $\prod_{\alpha \in I} A_\alpha \times \{a_\alpha : \alpha \in \Lambda \setminus I\}$ is almost dense in $\prod_{\alpha \in I} X_\alpha \times \{a_\alpha : \alpha \in \Lambda \setminus I\}$. This ensures that $f(D) = \{0\}$, completing the proof. □

Theorem 2.13. Let $\{Y_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces and fix $a = (x_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} Y_\alpha$. Then the set

$$E = \{(y_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} Y_\alpha : y_\alpha \neq x_\alpha \text{ for finitely many } \alpha\}$$

is dense in $\prod_{\alpha \in \Lambda} X_\alpha$. Hence, if $f = 0$ on E, then $f = 0$ on $\prod_{\alpha \in \Lambda} X_\alpha$, establishing that $\prod_{\alpha \in \Lambda} A_\alpha$ is almost dense in the product space. □

Connectedness and Almost Dense Sets

Theorem 2.14. If a topological space X contains a connected almost dense subset, then X is connected.

Proof. Let $A \subset X$ be connected and almost dense. Assume, for contradiction, that X is disconnected. Then there exists a continuous surjection $f : X \to \{0,1\}$. Let $Y = f^{-1}(0)$, so that $X \setminus Y = f^{-1}(1)$. Since A is connected, either $A \subset Y$ or $A \subset X \setminus Y$.

Case 1: If $A \subset Y$, then $f(A) = \{0\}$ but $f(X) \neq \{0\}$, contradicting the almost denseness of A.

Case 2: If $A \subset X \setminus Y$, define $g = 1 - f$. Then $g(A) = \{0\}$ but $g(X) \neq \{0\}$, again contradicting almost denseness.

Hence, X must be connected. □

Remark. The converse is not true; a connected space can contain disconnected almost dense subsets. For example, let $X = \mathbb{R}$ with the topology of all subsets containing 0. Then X is connected, but the subset $A = \{1,2\}$ is almost dense and disconnected.

Characterization of Almost Dense Sets via Cozero Sets

Theorem 2.15. A subset $A \subset X$ is almost dense if and only if it intersects every nonempty cozero set in X.

Proof. Suppose A is almost dense and let $U = X \setminus Z(f)$ be a nonempty cozero set. If $A \cap U = \emptyset$, then $f(A) = \{0\}$. By almost denseness, $f(X) = \{0\}$, contradicting $U \neq \emptyset$. Conversely, assume A intersects every nonempty cozero set. If A were not almost dense, there would exist $f \in C(X)$ with $f(A) = \{0\}$ but $f(X) \neq \{0\}$. Then $X \setminus Z(f)$ is a nonempty cozero set not intersecting A, a contradiction. □

Remark. This result is analogous to the classical property of dense sets, which intersect every nonempty open set. It naturally leads to the question: if A is almost dense in X and $U \subset X$ is a nonempty cozero set, is $A \cap U$ almost dense in U? The answer is not straightforward and requires further investigation.

Almost Separable Spaces

Definition 3.1. A topological space X is almost separable if it contains a countable almost dense subset.

Theorem 3.2. Every separable space is almost separable.

Remark. The converse is false, as illustrated below.

Example 3.3. Let $X = \mathbb{R}$ with the cocountable topology τ_c. Any countable subset A is not dense, since $X \setminus A$ is open and $A \cap (X \setminus A) = \emptyset$. However, Q, the set of rationals, is almost dense. Any continuous $f: X \to \mathbb{R}$ is constant, so if $f(Q) = \{0\}$, then $f(X) = \{0\}$. Therefore, X contains a countable almost dense subset and is almost separable.

Theorem 3.4. The finite product of almost separable spaces is almost separable.

Proof. Immediate from Corollary 2.11. □

Theorem 3.5. Let $Y \subset X$ be almost dense and almost separable as a subspace. Then X is almost separable.

Proof. Let A be a countable almost dense subset of Y. For any $f \in C(X)$ with $f(A) = \{0\}$, the restriction $f|_Y \in C(Y)$ satisfies $f|_Y(A) = \{0\}$. Since A is almost dense in Y, $f|_Y(Y) = \{0\}$. By the almost denseness of Y in X, $f(X) = \{0\}$. Therefore, A is a countable almost dense subset of X. □

Theorem 3.6. Let $\{X_\alpha: \alpha \in \Lambda\}$ be a family of almost separable spaces with $|\Lambda| = \mathfrak{c}$. Then $\prod_{\alpha\in\Lambda} X_\alpha$ is almost separable.

Proof. Let $A_\alpha \subset X_\alpha$ be countable almost dense subsets. Define bijections $f_\alpha: \mathbb{N} \to A_\alpha$ and construct $f_\Lambda: \mathbb{N}^\Lambda \to \prod_{\alpha\in\Lambda} A_\alpha$ by $f_\Lambda((n_\alpha)_{\alpha\in\Lambda}) = (f_\alpha(n_\alpha))_{\alpha\in\Lambda}$. This mapping is continuous and surjective. Since $\mathbb{N}^\Lambda$ is separable (hence almost separable), $\prod_{\alpha\in\Lambda} A_\alpha$ is almost separable. As it is almost dense in the full product, the product space is almost separable. □

Pseudocompactness and Baire Category–Type Theorem

Theorem 4.1. For any topological space X, the following statements are equivalent:

X is pseudocompact, meaning every real-valued continuous function on X is bounded.

If $\{F_n: n \in \mathbb{N}\}$ is a sequence of zero sets in X having the finite intersection property, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

If $\{U_n: n \in \mathbb{N}\}$ is a countable collection of cozero sets covering X, then a finite subcollection suffices to cover X.

This theorem provides multiple characterizations of pseudocompact spaces, connecting boundedness of continuous functions with the intersection behavior of zero sets and the covering properties of cozero sets.

Theorem 4.2. Let X be a topological space. For any nonempty cozero set U and any point $x \in U$, there exist a cozero set V and a zero set F such that

$$x \in V \subseteq F \subset U.$$

Proof. Since U is cozero, there exists a continuous function $f: X \to [0,1]$ such that $U = f^{-1}((0,1]) = X \setminus f^{-1}(\{0\})$. Because $x \in U$, we have $f(x) > 0$. Choose a small $\delta > 0$ so that $0 < f(x) - \delta < f(x)$. Define

$$V = f^{-1}((f(x) - \delta, 1]), F = f^{-1}([f(x) - \delta, 1]).$$

Clearly, $x \in V \subseteq F \subset U$. Here, V is a cozero set since $(f(x) - \delta, 1]$ is open in $[0,1]$, and F is a zero set because $[f(x) - \delta, 1]$ is closed in $[0,1]$. □

Theorem 4.3 (Baire Category–Type Theorem). Let X be a pseudocompact space, and let $\{U_n: n \in \mathbb{N}\}$ be a sequence of almost dense cozero sets in X. Then the intersection

$$\bigcap_{n=1}^{\infty} U_n$$

is a nonempty almost dense subset of X.

Proof. Let $D = \bigcap_{n=1}^{\infty} U_n$. We first show that $D \neq \emptyset$ and then prove that it intersects every nonempty cozero set.

Take any nonempty cozero set $V \subset X$ and pick $x \in V$. Since U_1 is almost dense, $V \cap U_1 \neq \emptyset$ and is itself a cozero set. Choose $x_1 \in V \cap U_1$. By Theorem 4.2, there exist a cozero set V_1 and a zero set F_1 such that

$$x_1 \in V_1 \subseteq F_1 \subset V \cap U_1.$$

Continuing recursively, for each $n \geq 1$, select $x_{n+1} \in V_n \cap U_{n+1}$ and construct cozero sets V_{n+1} and zero sets F_{n+1} satisfying

$$x_{n+1} \in V_{n+1} \subseteq F_{n+1} \subset V_n \cap U_{n+1}.$$

The sequence $\{F_n\}$ forms a nested collection of nonempty zero sets. By Theorem 4.1 and pseudocompactness of X,

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Since each $F_n \subseteq \bigcap_{k=1}^{n} U_k \cap V$, we conclude that $D \cap V \neq \emptyset$. As V was an arbitrary nonempty cozero set, D is almost dense. □

Remarks on Separability Properties

It is well-known that separability is not hereditary, as demonstrated by Niemytzky's plane. Similarly, almost separability is not hereditary; the Niemytzky plane also provides a counterexample in this context. We now summarize the relationships among various notions of separability. From standard results ([1, Section 1.2]):

Strong sequentially separable
$\Longrightarrow$ Sequentially separable
$\Longrightarrow$ Separable.

From Theorem 3.2 in this paper, every separable space is almost separable. Combining these results, we obtain the chain of implications:

Strong sequentially separable
$\Longrightarrow$ Sequentially separable
$\Longrightarrow$ Separable
$\Longrightarrow$ Almost separable.

This establishes the hierarchical relationship among different separability concepts, clarifying the position of almost separable spaces in the broader topology framework.

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